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ON THE $\mathbb{I}$-CONDITION

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ABSTRACT
In [5], XI, S. Shelah formulated a condition on forcing notions ($\mathbb{I}$-condition) which implies that the forcing it satisfies does not add reals. It was proved that, under some additional demands, this condition is preserved by revised countable support iterations. We are going to show that these demands can be weakened. A few examples of simple forcing notions that can iterate while preserving the -condition, and hence without adding reals, are presented.

In [4–6] the semiproper forcing, S-condition and $\mathbb{I}$-condition were defined in order to handle interations of the kind $\langle P_i, \underset{\sim}{Q}_i \mid i < \kappa \rangle$ preserving $\aleph_1$ (usually without adding new reals), collapsing all the cardinals $< \kappa$ to $\aleph_1$. It gives the possibility to obtain consistency results for $\aleph_2$. All results of this kind known to us can be obtained using semiproper forcing notions, but it requires large cardinal assumptions. In [5], XI the $\mathbb{I}$-condition was defined in order to get the equiconsistency. There was the following restriction on the iteration $\langle P_i, \underset{\sim}{Q}_i \mid i < \kappa \rangle$ for i an inaccessible cardinal less than κ: Q_i was required to satisfy the $\mathbb{I}_i$-condition with

$$I \in \mathbb{I}_i \tag{1}$$

i^+-complete or with I a normal ideal concentrated on ordinals of cofinality $> \aleph_0$ in V^{P_i}.

Adding this condition was a drawback, but it was satisfied by all the examples occurring there. However Gitik, for the application developed in [2], needed to have many inaccessibles in, e.g., Q_i satisfies the $\mathbb{I}_i$-condition, namely with normal ideals concentrated on ordinals of cofinality $\aleph_0$. However for those forcing a "strong $\mathbb{I}$-condition" holds:

$$\mathbb{I}_i = \{I_i\}, \tag{2}$$

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where I_i is a normal ideal on i and the function F witnessing the $\mathbb{I}_i$-condition for Q_i (see Definition 1) does not depend on w.

So Gitik proved that we can replace (1) by "for each inaccessible i, (1) or (2)" (and of course that the examples he had in mind, the three examples presented here, satisfied it).

However the general theorem was esthetically unsatisfactory. Then Shelah proved that really only i-completeness of $\mathbb{I}_i$ is needed. We present here this proof and a few examples of forcing notations which is a possible iterate preserving $\mathbb{I}$-condition.

We assume that the reader is familiar with chapters X and XI of [5] or with [4], [6]. We refer to [5] or [4] and [6] for the terminology and most of the definitions. We shall give here only the main one.

Definition 1 [1]. Let $\mathbb{I}$ be a set of monotone families (or simply a set of ideals). A forcing notion P satisfies the $\mathbb{I}$-condition if there is a function F, so that for every $\mathbb{I}$-tree T (i.e., a tree T s.t. every $\eta \in T$, $\mathrm{Suc}_T(\eta)$ is a positive set modulo for some $I \in \mathbb{I}$), if f is a function $f: T \to P$ satisfying

(a) $\nu < \eta$ implies $f(\nu) < f(\eta)$ and

(b) there are fronts $J_n\,(n < \omega)$ of T (J is a front if $\eta, \nu \in J$ implies that none of them is an initial segment of the other, and every $\eta \in \mathrm{Lim}\, T$ has an initial segment which is a member of J) such that every member of J_{n+1} has a proper initial segment belonging to J_n and $\eta \in J_n$ implies

$$\langle \mathrm{Suc}_T(\eta), I_\eta, \langle f(\eta \,\hat{}\, \alpha) \mid \alpha \in \mathrm{Suc}_T(\eta)\rangle\rangle = F(\eta, w[\eta], \langle f(\nu) \mid \nu < \eta\rangle),$$

where $w[\eta] = \{k < l(\eta) \mid \eta \restriction k \in \bigcup_{e<\omega} J_e\}$.

Then for every T', $T \leqq^* T'$ there is some $p \in P$ such that $p \Vdash$ "$\exists \eta \in \mathrm{Lim}\, T'$ such that $\forall k < \omega, f(\eta \restriction k) \in \underset{\sim}{G}$" where $\underset{\sim}{G}$ is the P-name of the generic subset of P.

Let us give first a few examples of forcing notions which are not included into Shelah's [5] scheme, but by Proposition 6 below it is possible to iterate them far enough preserving the $\mathbb{I}$-condition.

(1) $\mathrm{Nm}'(D) = \{T \mid T$ is a subtree of ${}^{\omega>}\omega_2$ s.t. above each $\eta \in T$ there is a D-splitting point, i.e. some $\nu > \eta$ so that $\mathrm{Suc}_T(\nu) \notin D\}$, where D is a normal ideal on $\aleph_2$. As in [5] it can be shown that $\mathrm{Nm}'(D)$ satisfies the $\mathbb{I}$-condition for every I s.t. $D \in \mathbb{I}$.

Let us present now two more complicated examples of such forcing notions. See [2] for their applications.

(2) Suppose for some κ we have a stationary subset S of it so that every α in S is an inaccessible cardinal. Suppose also that $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i \mid i < \kappa \rangle$ is a RCS iteration, $P_\kappa = R \lim \bar{Q}$, it does not add reals, for every inaccessible $\alpha \leqq \kappa$ P_α satisfies α-c.c., $\alpha = \aleph_2^{V[\mathring{P}_\alpha]}$ and every α in S and its successor become ordinals $< \aleph_2$ of cofinality ω in $V[\mathring{P}_{\alpha+1}]$. In $V[\mathring{P}_\kappa]$ let us define the forcing notion $P^*[S]$. It will be the set of all ω-closed subsets c of S so that for every limit point β of c, $c \cap \beta$ intersects every closed unbounded subset of β which belongs to $V[\mathring{P}_\beta]$ (or by β-c.c. of P_β it is enough that $c \cap \beta$ intersects every club subset of β in V). The ordering on $P^*[S]$ is an end extension.

PROPOSITION 2. *If κ is a weakly compact cardinal and S is a positive set in the weakly compact filter on κ (i.e., the normal filter generated by $\{\beta < \kappa \mid \langle V_\beta, \in, R \cap V_\beta \rangle \models \varphi(R \cap V_\beta)\}$, where φ is a Π^1_1-formula, $R \subseteq V_\kappa$ and $V_\kappa \models \varphi(R)$), then in $V[\mathring{P}_\kappa]$ the forcing $P^*[S]$ satisfies the $\mathbb{I}$-condition for any $\mathbb{I}$ so that $NS_\kappa \restriction S \in \mathbb{I}$ where $NS_\kappa \restriction S = \{A \subseteq \kappa \mid A \in V$ and $A \cap S$ is nonstationary$\}$ and for a forcing notion P, $\mathring{P}$ denotes its generic subset.*

REMARK. Since P_κ satisfies κ-c.c., every closed unbounded subset of $\kappa = \aleph_2^{V[\mathring{P}_\kappa]}$ in $V[\mathring{P}_\kappa]$ contains a closed unbounded subset of κ which belongs to V.

PROOF. By κ-c.c. the weakly compact filter on κ generates the normal filter $\mathscr{F}$ in $V[\mathring{P}_\kappa]$ on $\kappa = \aleph_2^{V[\mathring{P}_\kappa]}$.

$$\mathscr{F} \text{ is } \{X \in \mathscr{P}^{V[\mathring{P}_\kappa]}(\aleph_2) \mid \exists C \in \mathrm{Wc}_\kappa F \supseteq C\}.$$

Also for every stationary $D \in \mathscr{P}^{V[\mathring{P}_\kappa]}(\kappa)$, $\{\beta < \kappa \mid \beta$ is an inaccessible cardinal in V and $D \cap \beta$ is a stationary subset of $\beta = \aleph_2^{V[\mathring{P}_\beta]}$ in $V[\mathring{P}_\beta]\} \in \mathscr{F}$. See Baumgartner [1] for such results.

Now let us define the function F. F has to determine $\mathrm{Suc}(\eta)$, filter I_η and an element $f(\eta')$ of $P^*[S]$ for any immediate successor η' of η. Let

$$\mathrm{Suc}(\eta) = \{\eta \,\hat{}\, \langle \beta \rangle \mid \beta < \aleph_2,\ \beta > \max(f(\eta))\}.$$

Let $I_\eta = NS_\kappa \restriction S$ and

$$f(\eta \,\hat{}\, \langle \beta \rangle) = f(\eta) \cup \{\beta\}$$

for $\eta \,\hat{}\, \langle \beta \rangle \in \mathrm{Suc}(\eta)$.

Now let $\langle T, I \rangle$ be an $\mathbb{I}$-tree and f is a function from T to $P^*[S]$ so that

(a) $\nu < \eta$ implies $f(\nu) <_{P^*[S]} f(\eta)$ and

(b) there are fronts J_n ($n < \omega$) of T s.t. every member of J_{n+1} has a proper initial segment belonging to J_n and $\eta \in J_n$ implies

$$\langle \mathrm{Suc}_T(\eta), I_\eta, \langle f(\nu) \mid \nu \in \mathrm{Suc}_T(\eta) \rangle \rangle = F(\eta, f \restriction \{\nu \mid \nu \leqq \eta\}).$$

Let $T'^* \geqq T$. We shall find some $p \in P^*[S]$ s.t. $p \Vdash_{P^*[S]} (\exists \eta \in \lim T'$ such that $f(\eta \restriction k) \in \underset{\sim}{G})$, where $\underset{\sim}{G}$ is the canonical name of the generic subset of $P^*[S]$. As in [5], XI, Lemma 4.4, let T'', $T'' > T'$ be a subtree so that every point in T'' either belongs to some front J_n (and thus fits the demands on F) and is a splitting point, or it has exactly one immediate successor.

For $\eta \in T''$ let us denote by η' the successor of η of minimal height belongs to some front. Note that there is only one such η'. For this η', $\mathrm{Suc}_{T''}(\eta') \subseteq S$ is a stationary subset of $\aleph_2$.

Let, for $\eta \in T''$, C_η be $\{\beta \leqq \aleph_2 \mid \beta$ is an inaccessible cardinal in V and $\mathrm{Suc}_T(\eta') \cap \beta \cap S$ is a stationary subset of $\beta = \aleph_2^{V[\mathring{P}_\beta]}$ in $V[\mathring{P}_\beta]\}$. Then each C_η belongs to $\mathscr{F}$. Let $C = \{\beta < \aleph_2 \mid \forall \eta \in T''$ if, for every $k <$ height of η, $\eta \restriction k$ is a splitting point of T'' implies $\eta(k) < \eta$, then $\beta \in C_\eta\}$.

Claim. $C \in \mathscr{F}$.

Proof. Otherwise $\aleph_2 - C$ is an $\mathscr{F}$-positive set. Now let us use the normality of F. So there is $A_0 \subseteq \aleph_2 - C$ $\mathscr{F}$-positive s.t. for every $\beta_1, \beta_2 \in A_0$, η_{β_1} and η_{β_2} are on the same level in T''. Let us find $A_1 \subseteq A_0$ $\mathscr{F}$-positive so that for some $k_0 < k_1 < \cdots < k_{n-1} < \omega$, for every $\beta \in A_1$, $\eta_\beta \restriction k_0, \ldots, \eta_\beta \restriction k_{n-1}$ are all splitting points of η_β in T''. Now let $A_2 \subseteq A_1$ be $\mathscr{F}$-positive so that for any $\beta_1, \beta_2 \in A_2$, $m < n$, $\eta_{\beta_1}(k_m) = \eta_{\beta_2}(k_m)$. But then $\eta_{\beta_1} = \eta_{\beta_2} = \eta$ for $\beta_1, \beta_2 \in A_2$, since every point in T'' is a splitting point or has only one immediate successor. Hence $A_2 \cap C_\eta = \varnothing$. Contradiction.

Now let $\alpha \in C \cap S$. It exists since S is Wc_κ-positive and so $\mathscr{F}$-positive. Also $\alpha \in C_\eta$ for some $\eta \in T''$ (take for example η to be the first splitting point in T'') and so it is inaccessible in V.

In $V[\mathring{P}_{\alpha+1}]$, $\mathrm{cf}\,\alpha = \mathrm{cf}\,\alpha^+ = \aleph_0$, so we have a sequence $\langle \mathbf{C}_n \mid n < \omega \rangle$ so that

(a) $\mathbf{C}_n \in V$ and it is a club in α in $V[\mathring{P}_\alpha]$ (or in V; it does not matter since P_α satisfies α-c.c.).

(b) $\mathbf{C}_{n+1} \subseteq \mathbf{C}_n$.

(c) For every closed unbounded subset of $\alpha\,\mathbf{C} \in V[\mathring{P}_\alpha]$ there is some n so that $\mathbf{C}_n \subseteq \mathbf{C}$.

Since $V[\mathring{P}_{\alpha+1}] \vDash \mathrm{cf}\,(\alpha^+) = \mathrm{cf}\,\alpha = \aleph_0$ the set of all closed unbounded subsets of α in $V[\mathring{P}_\alpha]$ can be represented as $\bigcup_{n<\omega} B_n$, where $B_n \in V[\mathring{P}_\alpha]$ and the cardinality of B_n in $V[\mathring{P}_\alpha]$ is $\aleph_1$. Now since α is $\aleph_2$ in $V[\mathring{P}_\alpha]$, $\bigcap B_n = E_n$ is club in $V[\mathring{P}_\alpha]$. Since P_α satisfies α-c.c., by [1] every closed unbounded subset of α in $V[\mathring{P}_\alpha]$ contains some closed unbounded subset of α which belongs to V. Let us define $\mathbf{C}_0$ to be a club in V which is contained in E_0, $\mathbf{C}_1 \subseteq \mathbf{C}_0$ a club in V which is contained in E_1, and so on.

Let now η_0 be the first splitting point in T''. Since $\alpha \in C_{\eta_0}$, $S \cap \mathrm{Suc}_{T''}(\eta_0) \cap \alpha$ is a stationary subset of α in $V[\mathring{P}_\alpha]$. So there is some $\beta_0 \in S \cap \mathrm{Suc}_{T''}(\eta_0) \cap \alpha \cap \mathrm{C}_0$. Let $\eta_1 = \eta_0 {}^\frown \langle \beta_0 \rangle$. Then $f(\eta_1) = f(\eta_0) \cup \{\beta_0\}$.

By the definition of C, $\alpha \in C_{\eta'_1}$ (where η'_1 is the successor of η of minimal height which belongs to some front or the same is a splitting point). Hence $S \cap \mathrm{Suc}_{T''}(\eta'_1) \cap \alpha$ is a stationary subset of α in $V[\mathring{P}_\alpha]$. Pick some $\beta_1 \in S \cap \mathrm{Suc}_{T''}(\eta'_1) \cap \alpha \cap \mathrm{C}_1$. Let $\eta_2 = \eta'_1 {}^\frown \langle \beta_1 \rangle$. Then

$$f(\eta_2) = f(\eta'_1) \cup \{\beta_1\}.$$

Note that since η'_1 belongs to some front, $\mathrm{Suc}_{T''}(\eta'_1) \subseteq \{\eta'{}^\frown \langle \beta \rangle \mid \beta < \aleph_2,\ \beta > \max(f(\eta'_1))$ and $\beta \in S\}$. So $f(\eta_2) \in P^*[S]$ and $\max(f(\eta'_1)) < \alpha$. In such a way we define a branch η of T'' and a sequence of conditions $\langle f(\eta_n) \mid n < \omega \rangle$ so that $f(\eta_{n+1}) > f(\eta_n)$, $f(\eta_{n+1}) \cap \mathrm{C}_n \neq \varnothing$ and $\max f(\eta_n) < \alpha$. Let now $p = \bigcup_{n<\omega} f(\eta_n) \cup \{\alpha\}$. Since $\cup(\bigcup_{n<\omega} f(\eta_n)) = \alpha$, $p \in P^*[S]$. Now $p \Vdash_{P^*[S]} (\exists \eta \in T''$ such that $f(\eta \restriction k) \in \underset{\sim}{G})$. □

Suppose κ is an inaccessible s.t. $\{\alpha < \kappa \mid \alpha$ is weakly compact$\}$ is stationary. Then using $P^*[S]$ instead of $P[S]$ and the variant of Namba forcing for changing the cofinality of both α, α^+ to $\aleph_0$ instead of Nm′ in XI, 7.3 [5], it is possible to find a generic extension of L in which the following holds:

(*) There is a club $C \subseteq \aleph_2$ so that every $\alpha \in C$ is an inaccessible in L and for every δ a limit point of C, $C \cap \delta$ is not disjoint to any club of δ from L.

Question. What is the consistency strength of (*)?

In [3], using weakly compact cardinals a model is constructed with a stronger property: there is C as above, s.t. for every δ a limit point of C, $C \cap \delta$ is almost contained in any club of δ from L.

(3) Suppose we are in the same situation as in (2). Assume also that κ is a measurable and S belongs to a normal ultrafilter $\mathcal{U}$ over κ. Let us define in $V[\mathring{P}_\kappa]$ the forcing notion $P^*\{\mathcal{U}\}$. It will be the set of all pairs $\langle c, A \rangle$ so that (i) c is an ω-closed subset of κ; (ii) for every limit point β of c, $c \cap \beta$ intersects every closed unbounded subset of β, which belongs to $V[\mathring{P}_\beta]$ (or the same to V); (iii) $A \in \mathcal{U}$. The ordering on $P^*\{\mathcal{U}\}$ is defined as follows: $\langle c_1, A_1 \rangle \geqq \langle c_2, A_2 \rangle$ if c_1 is an end extension of c_2, $A_1 \subseteq A_2$ and $c_1 - c_2 \subseteq A_2$.

Proposition 3. *$P^*\{\mathcal{U}\}$ satisfies the $\mathbb{I}$-condition for any $\mathbb{I}$ so that $\mathcal{U} \in \mathbb{I}$.*

Proof. First note that since P_κ satisfies κ-c.c. and $\kappa = \aleph_2^{V[\mathring{P}_\kappa]}$, $\mathcal{U}$ generates a normal filter $\bar{\mathcal{U}}$ on $\aleph_2$ in $V[\mathring{P}_\kappa]$. $\bar{\mathcal{U}} = \{X \in \mathcal{P}^{V[\mathring{P}_\kappa]}(\aleph_2) \mid \exists Y \in \mathcal{U}, Y \subseteq X\}$. Since $\mathcal{U}$

contains the weakly compact filter on κ, for every stationary $D \in \mathcal{P}^{V[\mathring{P}_\kappa]}(\kappa)$ $\{\beta < \kappa \mid \beta$ is an inaccessible cardinal in V and $D \cap \beta$ is a stationary subset of $\beta = \aleph_2^{V[\mathring{P}_\beta]}$ in $V[\mathring{P}_\beta]\}$ contains some set from $\mathcal{U}$.

Let us define the function F. F has determined $\mathrm{Suc}(\eta)$, filter I_η and an element $f(\eta')$ of $P^*\{\mathcal{U}\}$ for any immediate successor η' of η. Let $I_\eta = \mathcal{U}$. If $f(\eta) = \langle c, A\rangle$, then let $\mathrm{Suc}(\eta) = \{\eta {}^\frown \langle\beta\rangle \mid \beta \in A$ and $\beta > \max(c)\}$ and $f(\eta {}^\frown \langle\beta\rangle) = \langle c \cup \{\beta\}, A\rangle$.

Now let $\langle T, I\rangle$ be a Ɪ-tree $f : T \to P^*\{\mathcal{U}_i\}$ satisfies (a), (b) from Proposition 2. Let J_n $(n < \omega)$, T'' and η' for $\eta \in T''$ be as in Proposition 2.

Let $\eta \in T''$ and $f(\eta') = \langle c, A\rangle$. We define $C_\eta = \{\beta \in A \mid \beta$ is inaccessible in V, $\beta \in S$ and $\mathrm{Suc}_T(\eta') \cap \beta$ is a stationary subset of $\beta = \aleph_2^{V[\mathring{P}_\beta]}$ in $V[\mathring{P}_\beta]\}$. Then since $\bar{\mathcal{U}}$ is normal, $\mathrm{Suc}_T(\eta')$ is $\bar{\mathcal{U}}$-positive and so it is stationary, $C_\eta \in \bar{\mathcal{U}}$. Let $C = \{\beta < \aleph_2 \mid \forall \eta \in T''$ if for every $k <$ height of η, $\eta \mid k$ is a splitting point of T'' implies $\eta(k) < \beta$, then $\beta \in C_\eta\}$.

As in Proposition 3, $C \in \bar{\mathcal{U}}$. Let $\alpha \in C$. Then $\alpha \in C_\eta$ for some $\eta \in T''$. So α is an inaccessible and $\alpha \in S$. So in $V[\mathring{P}_{\alpha+1}]$, $\mathrm{cf}\,\alpha = \mathrm{cf}\,\alpha^- = \aleph_0$ and hence there is a sequence $\langle \mathbf{C} \mid n < \omega\rangle$ that satisfies (a)–(c), see Proposition 2.

Let now η_0 be the first splitting point in T''. By the definition $\alpha \in C_{\eta_0}$. Let $\beta_0 \in \mathrm{Suc}_{T''}(\eta_0) \cap \alpha \cap \mathbf{C}_0$ and $\eta_1 = \eta_0 {}^\frown \langle\beta_0\rangle$. Suppose that $f(\eta_0) = \langle c_0, A_0\rangle$, then $\beta_0 \in A_0$ since η_0 belongs to a front and so $\mathrm{Suc}_{T''}(\eta_0) \subseteq A_0$. Then $f(\eta_1) = \langle c_0 \cup \{\beta_0\}, A_0\rangle$. By the definition of C, $\alpha \in C_{\eta_1'}$. So $\mathrm{Suc}_{T''}(\eta_1') \cap \alpha$ is a stationary subset of α in $V[\mathring{P}_\alpha]$. Let us pick some $\beta_1 \in \mathrm{Suc}_{T''}(\eta_1') \cap \alpha \cap \mathbf{C}_1$. Let $\eta_2 = \eta_1' \cap \langle\beta_1\rangle$.

$$f(\eta_1') = \langle c_1, A_1\rangle \geqq f(\eta_1) = \langle c_0 \cup \{\beta_0\}, A_0\rangle.$$

It implies that $A_1 \subseteq A_0$ and so $\beta_1 \in A_0$. Also $\cup c_1 < \beta_1 < \alpha$. Hence $f(\eta_2) = \langle c_1 \cup \{\beta_1\}, A_1\rangle \geqq \langle c_0 \cup \{\beta_0\}, A_0\rangle$, and so on. We obtain a branch η of T'' and an increasing sequence of conditions $\{\langle c_n \cup \{\beta_n\}, A_n\rangle \mid n < \omega\}$ so that $\beta_n \in \mathbf{C}_n$ and $\beta_n < \alpha$. Since $\alpha \in C_{\eta_n'}$ for $n > 0$, $\alpha \in A_n$ for every n (see the definition of $C_{\eta_n'}$). Hence $\alpha \in \bigcap_{n<\omega} A_n \in \bar{\mathcal{U}}$. Let $R \in \mathcal{U} R \subseteq \bigcap_{n<\omega} A_n$. Then $\langle \bigcup_n c_n \cup \{\alpha\}, R\rangle \in P^*\{\mathcal{U}\}$ and it is stronger than every $\langle c_n \cup \{\beta_n\}, A_n\rangle$. Now $\langle \bigcup_n c_n \cup \{\alpha\}, R\rangle \Vdash_{P^*\{\mathcal{U}\}} (\exists \eta \in \lim T''$ such that for every k, $f(\eta \restriction k) \in \underset{\sim}{G})$. □

The common property of those three examples is that the forcing notion there satisfies the Ɪ-condition, but for the ideals s.t. the set on which we forced in the previous stages is positive. We are going to show that it is possible to iterate such forcing notions preserving Ɪ-condition. First a definition.

DEFINITION 4 [5]. $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i \mid i < \alpha \rangle$ is suitable for $\langle \mathbb{I}_{i,j}, \lambda_{i,j}, \mu_{i,j} \mid \langle i, j \rangle \in W^* \rangle$ iff

(i) $W^* = \{\langle i, j \rangle \mid i < j < \alpha,\ i$ is not strongly inaccessible$\}$,
(ii) $\bar{Q}$ is an RCS iteration,
(iii) $P_{i,j} = P_j / P_i$ satisfies the $\mathbb{I}_{i,j}$-condition for $\langle i, j \rangle \in W^*$,
(iv) For every $I \in \mathbb{I}_{i,j}$, $\cup I$ is a regular cardinal, I is $\lambda^+_{i,j}$-complete, $\lambda_{i,j} < \cup I < \mu_{i,j}$, $|P_i| \leqq \lambda_{i,j}$, and $\lambda_{i,j} \geqq \aleph_2$,
(v) if $i(0) < i(1) < i(2) < \alpha$, $\langle i(0), i(1) \rangle \in W^*$, $\langle i(1), i(2) \rangle \in W^*$ then $(\forall \chi < \mu_{i(0),i(1)})[(\lambda_{i(1),i(2)})^\chi = \lambda_{i(1),i(2)}]$.

PROPOSITION 5. *Suppose* $\bar{Q} = \langle P_i, Q_i \mid i < \kappa \rangle$ *is suitable for* $\langle \mathbb{I}_{i,j}, \lambda_{i,j}, \mu_{i,j} \mid \langle i, j \rangle \in W^* \rangle$, κ *is strongly inaccessible,* $|\cup I| + \lambda_{i,j} + \mu_{i,j} < \kappa$ *for any* $\langle i, j \rangle \in W^*$ *and* $I \in \mathbb{I}_{i,j}$, $\underset{\sim}{Q}_\kappa$ *is a* P_κ*-name of a forcing notion satisfying the* $\mathbb{I}_\kappa$*-condition, and* $\mathbb{I}_\kappa$ *is* κ*-complete. Then* $P_\kappa * \underset{\sim}{Q}_\kappa$ *satisfies the* $\mathbb{I}$*-condition for* $\mathbb{I} = \bigcup_{i,j} \mathbb{I}_{i,j} \cup \mathbb{I}_\kappa$.

PROOF. Let $\underset{\sim}{F}_{i,j}$ be a P_i-name for a witness to "$P_{i,j}$ satisfies the $\mathbb{I}_{i,j}$-condition" for $i < j < \kappa$, i non-limit, and $\underset{\sim}{F}_\kappa$ a P_κ-name for "$\underset{\sim}{Q}_\kappa$ satisfies the $\mathbb{I}_\kappa$-condition". Let $A_i = \{m \in W \mid \exists k\ m = 2^{i+1} \cdot k,\ k$ is an odd number$\}$. Then $\langle A_i \mid < \omega \rangle$ is a partition of the even positive numbers into pairwise disjoint, infinite sets and $\min A_i = 2^{i+1}$. We shall add 0 to A_0. Let us define the function F for the $\mathbb{I}$-condition. So we should define $F(\eta, w, \langle f(\eta \restriction l) \mid l \leqq l(\eta) \rangle)$. The definition will be like those of [5].

Case 1. $|w|$ *is even.*

Let i be such that $|w| \in A_i$, $w^* = \{l \in w \mid |w \cap l| \in A_1\}$ and let $\beta_\eta(0) = 0$, and for l, $l(\eta) \geqq l$, $\beta_\eta(l+1) = \min\{\gamma + l \mid f(\eta \restriction l) \restriction \kappa \in P_\gamma\}$. Now we are going to apply the function $\underset{\sim}{F}_{\beta_\eta(i),\beta_\eta(i+1)}$. Let us apply it to the restriction $\tilde{\eta}$ of η. We define $\tilde{\eta} = \langle \delta_1, \ldots, \delta_n \rangle$ for $\eta = \langle \gamma_1, \ldots, \gamma_n \rangle$ as follows:

> Let $k_0 \in w$ or $k_0 = n$ be s.t. $|w \cap k_0| = \min A_i$, then put $\delta_l = \gamma_l$ for all $l \leqq k_0$. Now for k, $k_0 \leqq k < n$ let $\delta_{k+1} = \gamma_{k+1}$ if $k \notin w$, or $k \in w$ and $|w \cap k| \in A_j$ for $j \leqq i$. Otherwise let $\delta_{k+1} = 0$.

The reason for such a definition of $\tilde{\eta}$ will be clear from the continuation. Suppose

$$\underset{\sim}{F}_{\beta_\eta(i),\beta_\eta(i+1)}(\tilde{\eta}, w^*, \langle f(\eta \restriction l) \restriction [\beta_\eta(i), \beta_\eta(i+1)] \mid l \geqq l(\eta) \rangle) \text{ is } \langle \underset{\sim}{I}, \langle \underset{\sim}{r}_\nu \mid \nu \in \cup \underset{\sim}{I} \rangle \rangle.$$

Let us pick $q_\eta \in P_{\beta(i)}$,

$$q_\eta \Vdash_{P_{\beta(i)}} \text{“} \underset{\sim}{I} = I_\eta \text{”}.$$

Now we define F to be $\langle I_\eta, \langle f(\nu) \mid \nu \in \mathrm{Suc}_T(\eta)\rangle\rangle$, where $\mathrm{Suc}_T(\eta) = \{\eta^\frown\langle\alpha\rangle \mid \alpha < \bigcup I_\eta\}$ and $f(\eta) \cup q_\eta \cup \underset{\sim}{r}_\nu = f(\nu)$.

Case 2. $|w|$ *is odd.*

Let $\underset{\sim}{F}_\kappa(\eta, w^*\langle f(\eta \restriction l)(\kappa) \mid l \leqq l(\eta)\rangle) = \langle \underset{\sim}{I}_\eta, \underset{\sim}{r}_\nu \mid \nu \in \bigcup \underset{\sim}{I}_\kappa\rangle$, where $w^* = \{l \in \omega \mid |w \cap l| \text{ is an odd number}\}$. Choose $q_\eta \in P_\kappa$, $q_\eta \geqq f(\eta)\restriction \kappa$, $q_\eta \Vdash_{P_\kappa}$ "$\underset{\sim}{I}_\eta = I_\eta$" for some $I_\eta \in \mathbb{I}_\kappa$. Now we define

$$F(\eta, w, \langle f(\eta \restriction l) \mid l \leqq l(\eta)\rangle) = \langle I_\eta, \langle f(\eta) \cup q_\eta \cup \underset{\sim}{r}_\nu \mid \nu = \eta^\frown\langle\alpha\rangle, \alpha < \bigcup I_\eta\rangle\rangle.$$

Let us prove that such defined F satisfies Definition 1. Let $(T, \boldsymbol{I})$, J_n $(n < \omega)$, f and $\langle T', \boldsymbol{I}\rangle^* \geqq \langle T, \boldsymbol{I}\rangle$ be as in the definition. W.l.o.g. $\bigcup_{n<\omega} J_n$ is the set of splitting points of $(T, \boldsymbol{I})$. For notational simplicity assume $J_n = \{\eta \in T \mid l(\eta) = n\}$.

Claim 1. *For every* $\langle T'', \boldsymbol{I}\rangle^* \geqq \langle T', \boldsymbol{I}\rangle$ *there is* $p \in P_\kappa$, $p \Vdash (\boldsymbol{T}_\kappa = \{\eta \in T'' \mid f(\eta)\restriction\kappa \in \underset{\sim}{G}\}$ *contains an* ω*-branch).*

Proof. We need the following lemma from [5]. But first the definition.

Definition 6 [5]. For a subset A of T we define by induction on the length of η, $\mathrm{res}_T(A, \eta)$ for each $\eta \in T$. Let $\mathrm{res}_T(A, \langle\ \rangle) = \langle\ \rangle$. If $\mathrm{res}_T(A, \eta)$ is already defined then let, for every $\eta^\frown\langle\alpha\rangle \in \mathrm{Suc}_T(\eta)$, $\mathrm{res}_T(A, \eta^\frown\langle\alpha\rangle) = \mathrm{res}_T(A, \eta)^\frown\langle\alpha\rangle$ if $\eta \in A$ and $\mathrm{res}_T(A, \eta^\frown\langle\alpha\rangle) = \mathrm{res}_T(A, \eta)^\frown\langle 0\rangle$ otherwise.

Lemma 7 ([5]). *Let λ, μ be cardinals satisfying $\lambda^{<\mu} = \lambda$ and let (T, I) be a tree in which for each $\eta \in T$ either $|\mathrm{Suc}_T(\eta)| < \mu$ or I_η is λ^+-complete. Then for every function $H: T \to \lambda$ there exist T', $\langle T, I\rangle \leqq^* \langle T', I\rangle$ such that for η, $\eta' \in T'$, $\mathrm{res}_T(A, \eta) = \mathrm{res}_T(A, \eta')$ implies $H(\eta) = H(\eta')$, $\eta \in A$ iff $\eta' \in A$, and if $\eta \in A$, $\mathrm{Suc}_{T'}(\eta) = \mathrm{Suc}_T(\eta) = \mathrm{Suc}_{T'}(\eta') = \mathrm{Suc}_T(\eta')$, where $A = \{\eta \in T \mid |\mathrm{Suc}_T(\eta)| < \mu\}$.*

By a repeated use of Lemma 7 we can get T^*, $T' \leqq^* T^*$ such that, for every $\eta \in T^*$, if $l(\eta) = 2^{i+1}$, for some $i > 0$, or $l(\eta) = 0$, for $i = 0$ and $\eta_j \in T^*$, $\eta_j > \eta$ $(j \in 2)$ then

$$\mathrm{res}_{T^*}(A_\eta, \eta_1) = \mathrm{res}_{T^*}(A_\eta, \eta_2)$$

implies

(i) $f(\eta_1)\restriction \beta_\eta(i+1) = f(\eta_2)\restriction \beta_\eta(i+1)$;

(ii) $\eta_1 \in A_\eta$ iff $\eta_2 \in A_\eta$;

(iii) if $\eta_1 \in A_\eta$ then $\mathrm{Suc}_{T'}(\eta_1) = \mathrm{Suc}_{T^*}(\eta_1) = \mathrm{Suc}_{T^*}(\eta_2) = \mathrm{Suc}_{T'}(\eta_2)$;

where $A_\eta = \{\nu \in T' \mid \nu \leqq \eta \text{ or } \nu > \eta \text{ and } |\mathrm{Suc}_{T'}(\nu)| < \mu_{\beta_\eta(i), \beta_\eta(i+1)}\}$.

Note that for $i \in \omega$, if $\eta \in T'$ is of the length 2^{i+1} when $i > 0$, or it is $\langle\ \rangle$ when $i = 0$, then f on $\mathrm{Suc}_{T'}(\eta)$ is determined by $\underset{\sim}{F}_{\beta_\eta(i), \beta_\eta(i+1)}$ and we apply $\underset{\sim}{F}_{\beta_\eta(i), \beta_\eta(i+1)}$ the first time over η.

Let $T_{\langle\rangle} = \{\mathrm{res}_{T^*}(A_{\langle\rangle}, \eta) \mid \eta \in T^*\}$. The definition of T^* implies that $T_{\langle\rangle}$ is a $\mathbb{I}_{0,\beta_{\langle\rangle}(1)}$-tree. Let, for $\eta \in T^*$, $f_{\langle\rangle}(\mathrm{res}_{T^*}(A_{\langle\rangle}, \eta)) = f(\eta) \restriction \beta_{\langle\rangle}(1)$. Then $f_{\langle\rangle}$ is a well-defined function on $T_{\langle\rangle}$ into $P_{\beta_{\langle\rangle}(1)}$. Since we applied $F_{0,\beta_{\langle\rangle}(1)}$ enough times, there is a condition $q_{\langle\rangle} \in P_{\beta_{\langle\rangle}(1)}$ which forces "$\exists \eta \in \mathrm{Lim}\, T_{\langle\rangle}$ $\forall k < \omega f_{\langle\rangle}(\eta \restriction k) \in \underset{\sim}{G}$". Note that we apply $F_{0,\beta_{\langle\rangle}(1)}$ on the tree $T_{\langle\rangle}$. But as we defined F in Case 1, we are using $F_{0,\beta_{\langle\rangle}(1)}$ not on $\eta \in T$ but on its restriction $\tilde{\eta}$ which is now equal to $\mathrm{res}_{T^*}(A_{\langle\rangle}, \eta)$. Let G_0 be a generic subset of $P_{\beta_{\langle\rangle}(1)}$, $q_{\langle\rangle} \in G_0$ and $\eta[G_0] \in \mathrm{Lim}\, T_{\langle\rangle}$ be some branch in $T_{\langle\rangle}$ s.t. for every $k < \omega$, $f_{\langle\rangle}(\eta[G_0] \restriction k) \in G_0$. We define in $V[G_0]$ a tree $T_1 = \{\eta \in T^* \mid \mathrm{res}_{T^*}(A_{\langle\rangle}, \eta) < \eta[G_0]\}$.

For every $\eta \in T_1$, if $\eta \notin A_{\langle\rangle}$ then $\mathrm{Suc}_{T_1}(\eta) = \mathrm{Suc}_{T^*}(\eta)$. It follows from (iii) (see the definition of T^*). Let η_1 be some point from the fourth level in T_1. The fourth level is the level where $F_{\beta_{\langle\rangle}(1), \beta_{\eta_1}(2)}$ is applied the first time. As above we can define

$$T_{\eta_1} = \{\mathrm{res}_{T_1}(A_{\eta_1}, \eta) \mid \eta \geqq \eta_1, \eta \in T_1\}$$

and

$$f_{\eta_1}(\mathrm{res}_{T_1}(A_{\eta_1}, \eta)) = f(\eta) \restriction \beta_{\eta_1}(2).$$

Also there is $q_{\eta_1} \in P_{\beta_{\eta_1}(2)}$ which forces "$\exists \eta \in \mathrm{Lim}\, T_{\eta_1}$ $\forall k < \omega f_{\eta_1}(\eta \restriction k) \in \underset{\sim}{G}$". We can pick a generic subset G_1 of $P_{\beta_{\eta_1}(2)}$, $q_{\eta_1} \in G_1$ and a branch $\eta[G_1] \in \mathrm{Lim}\, T_{\eta_1}$ in T_{η_1} s.t. for every $k < \omega$, $f_{\eta_1}(\eta[G_1] \restriction k) \in G_1$. In the same way we can define T_2 to be the set of all $\eta \in T_1$ s.t. $\eta \geqq \eta_1$ and $\mathrm{res}_{T_1}(A_{\eta_1}, \eta) < \eta[G_1]$. It is possible to continue and define T_n and G_n for every $n < \omega$. Since we are using the revised countable support iterations, this process will give us some condition, namely $\bar{q} = \langle q_{\langle\rangle}, q_{\eta_1}, q_{\eta_2}, \ldots \rangle$ which forces that there is a branch $\eta \in \mathrm{Lim}\, T^*$ s.t. $\forall k < \omega$, $f(\eta \restriction k) \restriction \kappa \in \underset{\sim}{G}$. □

Suppose now that G is a generic subset of P_κ. Let us define, in $V[G]$, $T_\kappa[G] = T_\kappa^0[G] = \{\eta \in T' \mid (f(\eta) \restriction \kappa) \in G\}$, $T_\kappa^{\alpha+1}[G] = \{\eta \in T_\kappa^\alpha[G] \mid$ there is at least one successor of η in $T_\kappa^\alpha[G]$ and for every $l \leqq l(\eta)$, if on $\eta \restriction l$ F_κ was used, then $\mathrm{Suc}_{T_\kappa^\alpha[G]}(\eta \restriction l)$ is $I_{\eta \restriction l}$-positive, where $I_{\eta \restriction l}$ is defined by $F_\kappa\}$,

$$T_\kappa^\alpha[G] = \bigcap_{\beta < \alpha} T_\kappa^\beta[G] \qquad \text{for limit } \alpha.$$

For some $\alpha(G)$ large enough $T_\kappa^{\alpha(G)+1}[G] = T_\kappa^{\alpha(G)}[G]$.

If $T_\kappa^{\alpha(G)}[G] \neq \varnothing$ then, since F_κ was used on the fronts J_n $(n < \omega)$, there is some $q \in Q_\kappa$, $q \Vdash_{Q_\kappa}$ "$\exists \eta \in \mathrm{Lim}\, T_\kappa^{\alpha(G)}[G]$ $\forall k < \omega$, $f_\kappa(\eta \restriction k) \in \underset{\sim}{G}$", where $f_\kappa(\eta) =$

$f(\eta)(\kappa)$. So it is enough to show that if there is $p \in P_\kappa$ that forces "$T_\kappa^{\alpha(G)}[G] \neq \emptyset$" then $\langle p, q \rangle \in P_\kappa * Q_\kappa$ forces "$\exists \eta \in \operatorname{Lim} T'$ $\forall k < \omega$ $f(\eta \restriction k) \in G$".

Suppose otherwise. Then for every generic $G \subseteq P_\kappa$ and every $\eta \in T'$ so that F_κ was applied on η there is an ordinal $\alpha(\eta, G)$ and a set $C_\eta[G] \in I_\eta$ (in V), such that

$$\operatorname{Suc}_{T_\kappa^{\alpha(\eta,G)}[G]}(\eta) \subseteq C_\eta[G]$$

or

$$C_\eta[G] = \emptyset \qquad \text{in case } \operatorname{Suc}_{T_\kappa^{\alpha(\eta,G)}[G]}(\eta) \notin I_\eta$$

but for some $l < l(\eta)$, $\operatorname{Suc}_{T_\kappa^{\alpha(\eta,G)}[G]}(\eta \restriction l) \in I_{\eta \restriction l}$. Let C_η be a name for such a set. We define $C_\eta^* = \bigcup \{C \mid \exists p \in P_\kappa,\ p \Vdash C_\eta = \check{C}\}$. Since P_κ satisfies κ-c.c. this is the union in V of less than κ sets in I_η and hence $C_\eta^* \in I_\eta$. Let now

$$T'' = \{\eta \in T' \mid \text{there is no } l < l(\eta) \text{ so that } C_{\eta \restriction l}^* \text{ is defined and } \eta(l) \in C_{\eta \restriction l}^*\}.$$

For T'' let us define $T_\kappa[G]$, $T_\kappa^\alpha[G]$ as we did for T'. We shall denote these sets by $\bar{T}_\kappa[G]$ and $\bar{T}^\alpha[G]$. Now note that $T_\kappa^\alpha[G] \cap \bar{T}_\kappa[G] = \bar{T}_\kappa^\alpha[G]$ for every α, since always $C_\eta^* \in I_\eta$.

Claim 2. *For every α, $\bar{T}_\kappa^{\alpha+1}[G] = \{\eta \in \bar{T}_\kappa^\alpha[G] \mid \eta$ has at least one successor in $\bar{T}_\kappa^\alpha[G]\}$.*

Proof. If for some $\eta \in \bar{T}_\kappa^\alpha[G]$, F_κ was used on $\eta \restriction l$ $(l \leq l(\eta))$ and $\operatorname{Suc}_{T_\kappa^\alpha[G]}(\eta \restriction l) \in I_{\eta \restriction l}$, then $\operatorname{Suc}_{T_\kappa^\alpha[G]}(\eta \restriction l) \subseteq C_{\eta \restriction l}^*$ and hence $\operatorname{Suc}_{\bar{T}_\kappa^\alpha[G]}(\eta \restriction l) = \emptyset$. So $l = l(\eta)$ and η has now successors in $\bar{T}_\kappa^\alpha[G]$.

The claim implies that for every generic $G \subseteq P_\kappa$ in $\bar{T}_\kappa[G]$ there is no ω-branch, which contradicts Claim 1. □

Using Proposition 6, Conclusion 6.6 from [5] can be formulated as follows.

Proposition 8. *Suppose*

(a) $\bar{Q} = \langle P_i, Q_i \mid i < \alpha \rangle$ *is an RCS iteration,*

(b) Q *satisfies the* $\mathbb{I}_i$*-condition and* $\mathbb{I}_i$ *is* $\aleph_2$*-complete in* V^{P_i} *(but* $\mathbb{I}_i \in V$*),*

(c) *if* $\operatorname{cf}(i) < i \vee (\exists j < i) | P_j | \geq i$, *then for some* λ, μ, $\bigcup_{j \geq i} \mathbb{I}_j$ *is* λ^+*-complete,* $(\forall I \in \bigcup_{j<i} \mathbb{I}_j)(| \bigcup I | < \mu)$ *and* $\lambda = \lambda^{<\mu}$,

(d) *if* $\operatorname{cf} i = i \wedge (\forall j < i) | P_j | < i$ *then* $\mathbb{I}_i$ *is* i*-complete.*

Then $R \operatorname{Lim} \bar{Q}$ *satisfies the* $(\bigcup_{i<\alpha} \mathbb{I}_i)$*-condition.*

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